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Some Covered Graphs

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Abstract: **Let G = (V, E) be an undirected graph, k and t be positive integers and ∑ be the set of symbols. Then a feasible labeling is defined as an assignment of a set $L_v \subseteq \sum$ to each vertex v ϵ V, such that (i) $| L_v | \leq k$ for all v ϵ V and (ii) each label α ϵ ∑ is used no more than t times. In a feasible labeling, Commanality Index c(e) of an edge e = {i,j} is $| L_i \cap L_j |$. An edge e = {i,j} is said to be covered by a feasible labeling if c(e) ≥ 1 ,that is, if $|L_i \cap L_j| \neq \varphi$ A graph G is said to be a covered graph if there exists a feasible labeling that covers each edge e ϵ E. Hence a graph G is said to be covered if we have an assignment of at most k labels to each node of G such that each label is assigned to at most t nodes and there is at least one common label among the labels assigned at the two endpoints of each edge. In this paper we have proven that the graphs Prism (Dn), Corona ($CnOK_2$) are covered graphs with k =2 & t=4 whereas Wheel, Wn is a covered graph with k =2 & t ≥ 4.**

Index Terms: **covered graph, Corona graph, feasible labeling Prism, Wheel.**

1. Introduction

A graph G is a pair (V(G), E(G)) where V (G) is a nonempty finite set of elements known as vertices and E (G) is family of unordered pairs of elements of V (G) known as edges.

When the vertices or edges or both are assigned with integers or symbols using some conditions then we say that the graph is labeled. There are various labeling techniques. Some of them are feliticious labeling, harmonic labeling, prime labeling, Feasible labeling etc. In most of the papers, researchers have defined these labeling techniques using labeling functions.

In this paper we define algorithms for feasible labeling for some graphs and hence prove that those graphs are covered graphs.

R.Chandrasekaran,M.Dawande and M.Baysan [2] defined Feasible labeling and Covered graph as follows:

Let G = (V, E) be an undirected graph, k and t be positive integers and ∑ be the set of symbols. Then a feasible labeling is defined as an assignment of a set $L_v \subseteq \sum$ to each vertex v ϵ V, such that (i) $| L_v | \leq k$ for all v ϵ V and (ii) each label α ϵ ∑ is used no more than t times .In a feasible labeling , Commanality Index c(e) of an edge e = {i,j} is $| L_i \cap L_j |$. An edge e = {i,j} is said to be covered by a feasible labeling if c(e) ≥ 1 ,that is, if $|L_i \cap L_j| \neq \varphi$ A graph G is said to be a covered graph if there exists a feasible labeling that covers each edge e ϵ E. Hence a graph G is said to be covered if we have an assignment of at most k labels to each node of G such that each label is assigned to at most t nodes and there is at least one common label among the labels assigned at the two endpoints of each edge [2]. A graph G can be covered if $\deg_G(i) \leq k(t-1)$ for all i ϵ V. [2]

In this paper we have proven that three graphs, namely Prism, Wheel and Corona graph are covered graphs, for k = 2 , t ≥ 4 .We observe that for all these graphs $\deg_G(i) \leq k(t-1)$ for all i ϵ V.

The Cartesian product $C_n \times K_2$ of the cycle C_n ,n ≥ 3 and the complete graph of order 2 is called Prism graph D_n [1] . For every integer n ≥ 3, a wheel graph W_n is the graph defined by a pair of sets (V,E) , where V = { $c, v_0, v_1, \ldots, v_{n-1}$ } and E = { (c,v_i), (v_0,v_n), (v_i,v_{i+1}) | i = 0,1,….n-1 } The vertex c is called the centre of the wheel , each edge (c,v_i) , for i = 0,1,…. n-1, is called a spoke, and the cycle $C_n = W_n - c$ is called the rim [5].

The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph G obtained by taking one copy of G_1 (which has p_1 points) and p_1 copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2. We consider $C_n \circ K_2$

Most of the notations and terminologies are taken from the paper titled 'On a labeling problem in graphs' [2] and from Harary [3].

2. A covered graph Prism ,D_n for $n \geq 3$

In this section we give an algorithm for defining a feasible labeling for a prism graph, D_n for $n \geq 3$ and prove that is a covered graph for $k = 2$ and $t = 4$

2.1 Theorem : For $k = 2$ and $t = 4$, Prism graph D_n , $n \geq 3$ is a covered graph .

Proof : Let $v_1, v_2, \ldots, v_n$ be the vertices on inner cycle and let $u_1, u_2, \ldots, u_n$ be the vertices on outer cycle of D_n . Let $e_i = v_i u_i$, $1 \leq i \leq n$ denote the spoke of D_n . Let $\sum$ be the set of symbols, S be the set of completely labeled vertices and L be the set of labels used from $\sum$. Following is the algorithm which gives feasible labeling to $G = D_n$, $n \geq 3$

<u>2.2 Algorithm 1. Feasible Labeling to D_n</u>

INPUT: Graph $D_n = (V, E)$

OUTPUT : Graph D_n with set S of labelled vertices.

1. $L \leftarrow \sum$; $S \leftarrow \varphi$
2. $v_1 \leftarrow a_1, a_2$; $u_1 \leftarrow a_1, a_n$; $v_n \leftarrow a_1$
3. $S \leftarrow S \cup \{ v_1, u_1 \}$
4. $i \leftarrow 2$
5. while $i < n$
   $V_i \leftarrow a_i , a_{i+1}$; $u_i \leftarrow a_{i-1} , a_i$
6. $S \leftarrow S \cup \{ v_i, u_i \}$; $L \leftarrow L \setminus \{ a_{i-1} \}$
7. $i \leftarrow i + 1$
8. $v_n \leftarrow a_n$; $u_n \leftarrow a_{n-1} , a_n$
9. . $S \leftarrow S \cup \{ v_n, u_n \}$; $L \leftarrow L \setminus \{ a_{n-1}, a_n \}$
10 End

2.3 Remark : From this algorithm it is clear that each vertex gets 2 labels

Viz. $v_i = (a_i , a_{i+1}) = u_{i+1}$, $1 \leq i < n$

and $v_n = (a_1, a_n) = u_1$

and each label is used exactly 4 times .

Also $c(e) = 1$

Hence Prism is a covered graph for $n \geq 3$

2.4 Illustration : Figure 1 shows a covered graph D_4

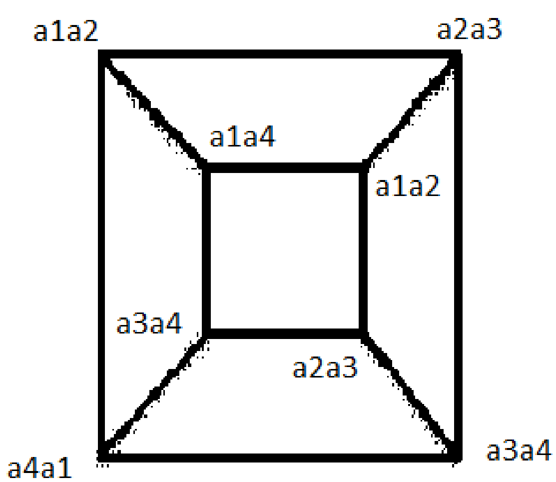

Figure 1

3. A covered graph Wheel, Wn for $n \geq 5$

In this section we prove that a Wheel graph, W_n for $n \geq 5$ is a covered graph for $k = 2$ and $t \geq 4$

3.1 Theorem

For $k = 2$ and $t \geq 4$, Wheel , W_n , $n \geq 5$ is a covered graph.

Proof : Let $v_1, v_2, \ldots, v_n$ be the vertices on the rim and u be the center vertex of the wheel W_n.

Let $\sum = \{a,b,c,d\}$

We consider two cases for even n and odd n.

Following algorithm gives feasible labeling to W_n, where n is odd. Let $n = 2m - 1$, $m \geq 3$

<u>3.2 Algorithm 2. Feasible labeling to W_n for odd n</u>

Input : Graph $W_n = (V, E)$

Output : Graph W_n with set S of labeled vertices.

1. $L \leftarrow \sum$; $S \leftarrow \varphi$
2. $u \leftarrow a, b$
3. $S \leftarrow S \cup \{ u \}$
4. for $i = 1$ to m
5. $v_i \leftarrow a$
6. $S \leftarrow S \cup \{ v_2, \ldots, v_{m-1} \}$
7. for $i = m + 1$ to $2m-1$
8. $v_i \leftarrow b$
9. $S \leftarrow S \cup \{ v_{m+2}, \ldots v_{2m-2} \}$
10. v_1 , v_m , v_{m+1} , $v_{2m-1} \leftarrow c$
11. $S \leftarrow S \cup \{ v_1, v_m , v_{m+1} , v_{2m-1} \}$, $L \leftarrow L \setminus \{ a,b,c \}$
12. End

Following algorithm gives feasible labeling to W_n, where n is even. Let $n = 2m$, $m \geq 3$

<u>3.3 Algorithm 4. Feasible labeling to W_n for even n</u>

Input : Graph $W_n = (V, E)$

Output : Graph W_n with set S of labelled vertices.

1. $L \leftarrow \sum$; $S \leftarrow \varphi$
2. $u \leftarrow a, b$
3. $S \leftarrow S \cup \{ u \}$
4. for $i = 1$ to m
5. $v_i \leftarrow a$
6. $S \leftarrow S \cup \{ v_2, \ldots, v_{m-1} \}$
7. for $i = m + 1$ to $2m$
8. $v_i \leftarrow b$
9. $S \leftarrow S \cup \{ v_{m+2}, \ldots v_{2m-1} \}$
8. v_1 , v_m , v_{m+1} ,$v_{2m} \leftarrow c$
9. $S \leftarrow S \cup \{ v_1 , v_m , v_{m+1} , v_{2m} \}$,$L \leftarrow L \setminus \{ a,b,c \}$
10. End

3.4 Remark : For $n = 2m-1$ or $2m$, $t = m+1$. And each label is used at the most t times.

3.5 Illustration : Figure 2 shows a covered graph W_6

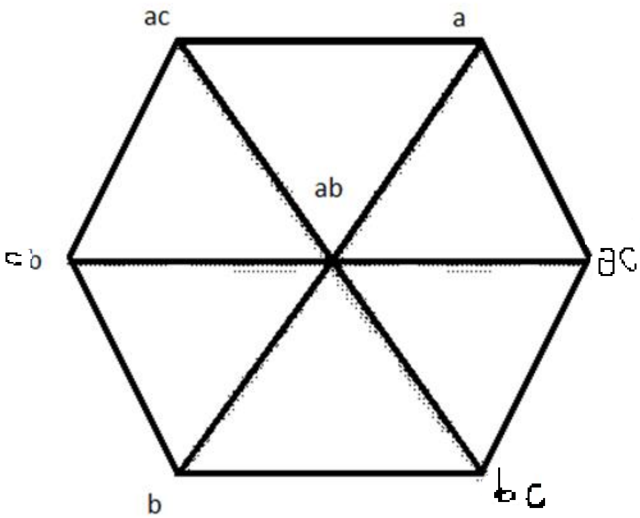

Figure 2

4. A corona graph C_n o K_2, $n \geq 3$

In this section we prove that a corona graph C_n o K_2 for $n \geq 3$ is a covered graph for k = 2 and t = 4

4.1 Theorem

For k = 2 and t = 4 , a corona graph C_n o K_2 ,$n \geq 3$ is a covered graph.

Proof : Let $v_1, v_2, \ldots, v_n$ be the vertices on the cycle C_n and v_i' , v_i'' be the vertices on C_3, adjacent to v_i for all i = 1ton

Let ∑ be the set of symbols $a_1, a_2, \ldots a_n$, , S be the set of completely labeled vertices and L be the set of labels used from ∑. Following is the algorithm which gives feasible labeling to G = C_n o K_2 , $n \geq 3$

4.2 Algorithm 4. Feasible labeling to C_n o K_2

Input : Graph C_n o K_2 = (V, E)

Output : Graph C_n o K_2 with set S of labelled vertices.

1. L ← ∑ ; S ← φ
2. i ← 1
3. while i < n+1
4. v_i , v_i' , v_i'', v_{i+1} ← a_i
5. S ← S U { v_i' , v_i'' } ; L ← L \ { a_i }
6. i ← i + 1
7. S ← S U { $v_2, \ldots, v_n$ }
8. v_1 ← a_n
9. S ← S U { v_1 }
10. End

4.3 Remark : From this algorithm it is clear that each vertex on C_n gets 2 labels and v_i' , v_i''get one label.

Viz. $v_i = (a_{i-1} , a_i)$, $1 < i \leq n$

and $v_1 = (a_1, a_n)$

and each label is used exactly 4 times.

Also c(e) = 1

Hence C_n o K_2 is a covered graph for $n \geq 3$

4.4 Illustration : Figure 3 shows a covered graph C_5 o K_2

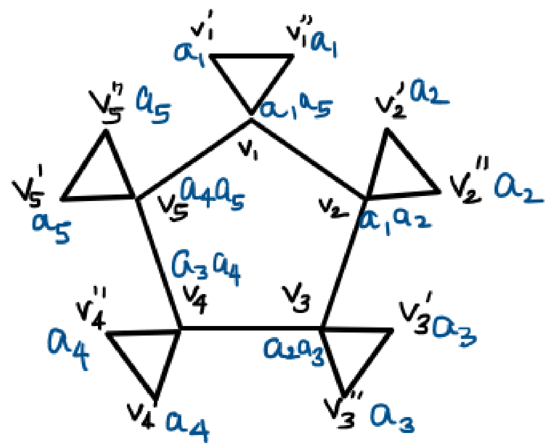

Figure 3

Conclusion

In this paper it is shown that some graphs are covered graphs, by defining the algorithms. For future work we plan to work on applications of these results in Software Programming. To investigate similar results for other graph families is an open area of research.

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